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Critical exponents of the dilute Ising model from four-loop expansions

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Abstract. Four-loop expansions of the coefficients of the Callan–Symanzik equation for 3D and 2D dilute Ising models are calculated. Fixed-point coordinates and critical exponents are estimated by two methods of summation of double series: a generalisation of the Padé–Borel approximation and the first confluent form of the ε algorithm of Wynn. Summation of the double series for the 2D dilute Ising model gives exponents very close to exponents of the pure Ising model, in accordance with the exact solution. The two methods are also applied to summation of single-variable series of pure Ising and polymer models, both in 3D and 2D cases. Both methods are shown to provide close agreement between present estimates and results obtained earlier either exactly in conformal invariant theories (2D) or numerically with high accuracy (3D). An additional test is provided by estimation of critical exponents of the 2D O(n) model for $n = -1$. The methods tested this way are used to calculate critical exponents of the 3D dilute Ising model. The values obtained are consistent with recent experimental results.

1. Introduction

The critical behaviour of physical systems described by the quenched site-diluted Ising model or by a random exchange model (REIM) is a subject of increasing interest. The general features of the transition have been elucidated by Harris and Lubensky (1974), Khmelnitsky (1975) and Lubensky (1975). Experimental studies of three-dimensional random Ising magnets have revealed sharp phase transitions with critical exponents clearly different from those of the pure Ising systems (for a review of relevant experiments see Thurston *et al* (1988)). The experimental data are consistent with recent theoretical predictions. The latter have been obtained first by Newman and Riedel (1982) by the scaling field method and then by a generalisation of the Padé–Borel technique (Jug 1983, Mayer and Sokolov 1984) which implies replacement of Padé approximants (PA) under the Laplace integral with their two-variable counterparts. A possible choice for such a replacement is an approximation invented by Chisholm (1973). This Chisholm–Borel method has been used to find the 3D REIM exponents from three-loop (Mayer and Sokolov 1983) and four-loop expansions (Mayer *et al* 1988).

In this paper four-loop expansions of the coefficients $\beta_u(u, v)$, $\beta_v(u, v)$, $\eta(u, v)$ and $\eta_4(u, v)$ of the Callan–Symanzik equation are presented. To extract critical properties of 3D and 2D dilute models from these expansions two different methods of summation of divergent double series are used: (i) a Padé–Borel approximation and (ii) the 'continuous prediction' method realised as the first confluent form of the ε algorithm of Wynn (Brezinski 1977, Lovitch and Marziani 1983, Marziani 1984). In case (i) the PA are constructed for the series in powers of an auxiliary variable λ with coefficients made up of sums of terms of the fixed order in the initial series, e.g. the coefficient of λ^2 is $(a_{20}u^2+a_{11}uv+a_{02}v^2)$. Putting $\lambda=1$ at the end gives a Padé approximation to the initial series labelled $[L/M]$ with L and M being the powers of λ in the nominator and denominator of the PA, respectively. The choice of the PA in this form removes ambiguities in the calculation of coefficients of the denominator, unavoidable in the Chisholm approximation (Chisholm 1973). Method (ii) applied to the Mittag-Leffler transform of the initial series produces a sequence of the same two-variable PA in a limiting case. This relationship allows one to regard methods (i) and (ii) as two distinct ways of improvement of convergence for different sequences in the Padé table.

The outline of this paper is as follows. In § 2 necessary details of the computation of renormalisation group (RG) functions as well as four-loop expansions for 3D and 2D REIM are given. Section 3 is devoted to essential aspects of summation methods. Section 4 deals with numerical results and § 5 contains conclusions.

2. The model and RG expansions

In this section key points of the RG treatment of REIM are sketched (for details see, e.g., Grinstein and Luther 1976).

The REIM Hamiltonian is thermodynamically equivalent to the n-component Hamiltonian of the Heisenberg model with cubic anisotropy in the limit $n\to 0$:

$$H=\frac{1}{2}\int \mathrm{d}^dx\left[\sum_{i=1}^{n}(\nabla\varphi_i)^2+m_0^2\sum_{i=1}^{n}\varphi_i^2+\frac{u_0}{12}\left(\sum_{i=1}^{n}\varphi_i^2\right)^2+\frac{v_0}{12}\sum_{i=1}^{n}\varphi_i^4\right] \tag{2.1}$$

where $m_0^2\sim T-T_c(p)$ and $u_0=u_0(p)<0$ is the bare coupling constant of effective interaction of fluctuations due to the presence of impurities, with $(1-p)$ being the concentration of non-magnetic impurities in the magnetic lattice. Calculations are for arbitrary n and the quenching of impurities is incorporated by taking the limit $n\to 0$ at the end.

Impose the zero-moments renormalisation conditions (Brézin *et al* 1976, Grinstein and Luther 1976) for conventionally defined two-point and four-point, amputated, single-particle irreducible vertex functions $\Gamma_R^{(2)}$, $\Gamma_{R(u)}^{(4)}$ and $\Gamma_{R(v)}^{(4)}$:

$$\begin{aligned}
&\Gamma_R^{(2)}(p,-p;m,u,v)|_{p^2=0}=m^2\\
&\frac{\mathrm{d}}{\mathrm{d}p^2}\Gamma_R^{(2)}(p,-p;m,u,v)|_{p^2=0}=1\\
&\Gamma_{R(u)}^{(4)}(p_i;m,u,v)|_{p_i=0}=m^\varepsilon u\\
&\Gamma_{R(v)}^{(4)}(p_i;m,u,v)|_{p_i=0}=m^\varepsilon v
\end{aligned} \tag{2.2}$$

with one more condition for the φ^2 insertion:

$$\Gamma_R^{(1,2)}(p;q,-q;m,u,v)|_{p=q=0}=1.$$

From renormalisation conditions (2.2) follow the double expansions for the renormalisation constants Z_3 for the field φ, Z_{1u} and Z_{1v} for the vertices and Z_4 for the φ^2 insertion. The coefficients of the Callan-Symanzik equation are then defined by the set of equations

$$
\begin{aligned}
\beta_u(u, v) &= \left.\frac{\partial u}{\partial \ln m}\right|_{u_0, v_0} \\
\beta_v(u, v) &= \left.\frac{\partial v}{\partial \ln m}\right|_{u_0, v_0} \\
\eta(u, v) &= \left.\frac{\partial \ln Z_3}{\partial \ln m}\right|_{u_0, v_0} \\
\eta_4(u, v) &= \left.\frac{\partial \ln Z_4}{\partial \ln m}\right|_{u_0, v_0}
\end{aligned}
\tag{2.3}
$$

where $u_0 = m^{\varepsilon} u Z_{1u} Z_3^{-2}$ and $v_0 = m^{\varepsilon} v Z_{1v} Z_3^{-2}$.

The computation of Feynman graphs contributing to equations (2.2) is reduced to the simple task of obtaining n-component field factors for the vertices involved, because the momentum integrals and symmetry factors are known (Nickel *et al* 1977). The resulting expansions are summarised in tables 1 and 2. The series are normalised by a change of variables $u \to u/(n+8)$, $v \to v/3$ so that the coefficients of the terms u, u^2 and v, v^2 become -1 and $+1$, respectively.

Table 1. RG functions of the 3D REIM. (mn) stands for the coefficient of the term $u^m v^n$.

(mn)	β_u/u	β_v/v	η	η_4
(00)	-1	-1		
(10)	1	$\frac{3}{2}$		$-\frac{1}{4}$
(01)	$\frac{2}{3}$	1		$-\frac{1}{3}$
(20)	$-\frac{95}{216}$	$-\frac{185}{216}$	$\frac{1}{108}$	$\frac{1}{16}$
(11)	$-\frac{50}{81}$	$-\frac{104}{81}$	$\frac{2}{81}$	$\frac{1}{6}$
(02)	$-\frac{92}{729}$	$-\frac{308}{729}$	$\frac{8}{729}$	$\frac{2}{27}$
(30)	0.389 922 6898	0.916 667 9106	0.000 771 3750	−0.035 767 2729
(21)	0.857 363 7982	2.132 996 3701	0.003 085 5001	−0.143 069 0917
(12)	0.467 388 5589	1.478 058 1216	0.003 085 5001	−0.146 667 9876
(03)	0.090 448 9508	0.351 069 5980	0.000 914 2223	−0.044 310 2531
(40)	−0.447 316 0963	−1.228 684 6326	0.001 589 8706	0.034 374 8466
(31)	−1.343 861 1316	−3.899 273 1308	0.008 479 3098	0.183 332 5150
(22)	−1.221 301 3925	−4.236 958 6776	0.013 245 6140	0.288 298 9658
(13)	−0.476 233 5809	−2.033 085 8783	0.008 083 0029	0.175 845 7491
(04)	−0.075 446 6920	−0.376 526 8279	0.001 796 2229	0.039 519 5688

It is easily seen from tables 1 and 2 that, if projected on one of the variables, any of the series represents the corresponding quantity of the pure Ising model or polymer ($n = 0$ limit) model. Both series are Borel summable. Critical exponents from single-variable expansions have recently been calculated to within four decimal places through the conformal mapping summation method (Le Guillou and Zinn-Justin 1980). However, the latter is based on subtle details of the asymptotics which are not known for the double expansions under consideration. At the same time earlier estimates of exponents derived from a Padé–Borel analysis (Baker *et al* 1978) are close to the values now believed to be exact. So it seems worth trying to obtain reasonable estimates of 3D REIM critical exponents by some generalisation of the Padé–Borel approximation.

Table 2. RG functions of the 2D REIM.

(mn)	β_u/u	β_v/v	η	η_4
(00)	-1	-1		
(10)	1	$\frac{3}{2}$		$-\frac{1}{2}$
(01)	$\frac{2}{3}$	1		$-\frac{2}{3}$
(20)	−0.744 9227	−1.448 1787	0.028 6589	0.210 9766
(11)	−1.048 7858	−2.173 9954	0.076 4238	0.562 6048
(02)	−0.216 0804	−0.716 1736	0.033 9661	0.250 0466
(30)	1.024 1730	2.399 6662	−0.001 7053	−0.188 8877
(21)	2.262 7012	5.607 5358	−0.006 8261	−0.755 5508
(12)	1.251 3563	3.927 9375	−0.006 8261	−0.773 7783
(03)	0.231 5657	0.930 7664	−0.002 0226	−0.233 5882
(40)	−1.853 2980	−5.059 3404	0.010 1404	0.277 1106
(31)	−5.609 5330	−16.158 7403	0.054 0823	1.477 9232
(22)	−5.165 6134	−17.731 1960	0.084 4413	2.335 4310
(13)	−2.006 9852	−8.550 2770	0.051 2689	1.439 4247
(04)	−0.311 6955	−1.582 3883	0.011 3931	0.323 0886

3. Methods of summation

Let a physical quantity of interest be represented by a double series

$$A(u, v) = \sum_{ij} a_{ij} u^i v^j \tag{3.1}$$

where $a_{ij} \to (i+j)!$ for $i, j \to \infty$ and $A(u, v)$ denotes any of the expansions in tables 1 and 2. These double series will be assumed to be Borel summable.

(i) Introduce a 'resolvent' series (Baker and Graves-Morris 1981)

$$F(u, v; \lambda) = \sum_{l=0}^{\infty} \lambda^l \sum_{k=0}^{l} u^k v^{l-k} a_{k,l-k} = \sum_{l=0}^{\infty} \lambda^l A_l \tag{3.2}$$

with the additional constraint $a_{k,l-k} = 0$ for $l < k$ and the obvious notation for A_l. Now the right-hand side of equation (3.2) is a series in powers of λ with coefficients A_l and it is possible to construct a sequence of PA $[L/M]$ to the order allowed by the conditions $L + M = 4$ and $L \geqslant M$, since the evaluation of critical exponents implies determination of the roots of functions $\beta_u(u, v)$ and $\beta_v(u, v)$. The sum of the series is then approximated by

$$\tilde{A}(u, v) = [L/M]|_{\lambda=1}. \tag{3.3}$$

These approximants hold the projection property of the initial series: if u or v is put equal to zero, equation (3.3) reduces to the conventional single-variable PA. The same procedure will be used for analytical continuation of the Borel-transformed series:

$$B(x, y) = \sum_{ij} \frac{a_{ij}}{(i+j)!} x^i y^j \tag{3.4}$$

in the Borel sum defined by

$$\tilde{A}(u, v) = \int_0^{\infty} \exp(-z) B(uz, vz)\, dz. \tag{3.5}$$

(ii) Another summation method is the first confluent form of the ε algorithm of Wynn (Brezinski 1977, Lovitch and Marziani 1983, Marziani 1984) applied to the sum formally represented by an integral (Sansone and Gerretsen 1960)

$$\tilde{A}(u, v) = \int_0^\infty \exp(-z) B_M(uz^\beta, vz^\beta)\, dz \tag{3.6}$$

with B_M being the Mittag-Leffler transform:

$$B_M(x, y) = \sum_{ij} \frac{a_{ij}}{\Gamma[\beta(i+j)+1]} x^i y^j. \tag{3.7}$$

For $\beta = 1$, expressions (3.6) and (3.7) give the Borel sum defined by equations (3.4) and (3.5). A better choice here is $\beta = 2$, as is argued below.

The evaluation of the sum (3.6) amounts to the following. Introduce a function

$$S(u, v; t) = \int_0^t \exp(-z) B_M(uz^\beta, vz^\beta)\, dz$$

so that

$$\tilde{A}(u, v) = \lim_{t\to\infty} S(t) \qquad S(t) \equiv S(u, v; t). \tag{3.8}$$

The limit (3.8) could be replaced by the limit of a convergent ε sequence (Brezinski 1977)

$$\lim_{t\to\infty} S(t) = \lim_{m\to\infty} \varepsilon_{2m}(t_0)$$

$$\varepsilon_{2m}(t_0) = H^{(0)}_{m+1}(t_0)/H^{(2)}_m(t_0) \qquad m = 0, 1, 2, \ldots \tag{3.9}$$

where t_0 is an arbitrary value of the variable t and $H^{(k)}_m(t)$ are Hankel functional determinants:

$$\begin{vmatrix} S^{(k)}_{(t)} & S^{(k+1)}_{(t)} & \cdots & S^{(k+m-1)}_{(t)} \\ & & & \\ S^{(k+m-1)}_{(t)} & S^{(k+m)}_{(t)} & \cdots & S^{(k+2m-2)}_{(t)} \end{vmatrix}. \tag{3.10}$$

To start the iterative process in the formula (3.9) one needs initial conditions. These are $H^{(k)}_0 = 1$, $H^{(k)}_1 = S^{(k)}$, $S^{(-1)}_{(t)} = 0$, $S^{(0)}_{(t)} = S(t)$. The functions $S^{(k)}_{(t)}$ in equation (3.10) are derivatives of $S(t)$ with respect to t:

$$S^{(k+1)}_{(t)} = \exp(-t) \sum_{m=0}^{N} (-1)^{k-m} C^m_k \frac{d^m B_m}{dt^m} \qquad k = 0, 1, 2, \ldots .$$

It is known that, for $\beta = 1$, the $\varepsilon_{2m}(t_0 = 0)$ sequence reproduces the PA sequence $[m-1/m]$ (Marziani 1983). The proof concerns the single-variable case, but is simply generalised to more than one variable. It turns out that, for the double series, the emerging approximants are given just by equation (3.3) for $L < M$. Thus the ε algorithm is related to the two-variable PA to the resolvent series in the limiting case $t_0 = 0$. However, the PA in the sequence $[m-1/m]$ have the degree of numerator less than that of the denominator, which makes them inappropriate for the evaluation of the zeros of $\beta_u(u, v)$ and $\beta_v(u, v)$. On the other hand, if $\beta = 2$, equation (3.9) generates a different PA sequence: [0/0], [0/1], [1/1], [1/2], [2/2], ..., i.e. the first convergents of the continued fraction constructed from the series (3.1) (Baker and Graves-Morris 1981). This is how the diagonal PA enter the scene to give a better convergent scheme.

As an aside, the first confluent form of the ε algorithm of Wynn can be viewed as a kind of perturbative improvement of convergence of the underlying PA sequence, the latter appearing at $t_0=0$. Varying the parameter t_0 it is possible to accelerate convergence of the ε sequence so it is highly desirable, especially for comparatively short initial series, to have as a 'zero-order approximation' the sequence including diagonal PA; hence the motivation to take $\beta=2$. In addition, $\beta=2$ provides better convergence for the ε sequence, as has been shown for the Rayleigh-Schrödinger perturbation series in the anharmonic oscillator (Marziani 1984) and Yukawa and funnel-like potential (Mayer 1988) cases.

The parameter t_0 is arbitrary only if an infinite number of terms is known. For a series of finite length t_0 is chosen to achieve fast convergence of the ε sequence (Marziani 1984, Mayer 1988). Since the REIM series are rather short, it is hardly possible to compare rates of convergence for different t_0. Instead I use the projection property mentioned above and take the value t_0 at which Ising ($u=0$) and polymer ($v=0$) critical exponents are as close as possible to their exact values. The fixed-point (FP) coordinates and critical exponents of the REIM are then calculated for this t_0. The REIM exponents are believed to be in the same error range as the corresponding values in the pure systems.

4. Numerical results

To reduce the uncertainty in exponent values the summation procedures are applied to the expansions $\eta(u, v)$ and $\eta_4(u, v)$ from tables 1 and 2, and also to the expansions obtained from these through scaling relations, e.g. $\nu(u, v)=[2+\eta_4(u, v)-\eta(u, v)]^{-1}$. As is well known, a pair of critical exponents is sufficient for calculation of the whole set of exponents in the given universality class. From numerical values of different pairs of exponents, say γ and ν or γ and η, etc, the rest is calculable by scaling relations. The results are centred at a certain value which is assumed as the final answer. Apart from the random FP ($u^*<0, v^*>0$), the set of RG equations produces polymer FP ($u^*>0, v^*=0$) and Ising FP ($u^*=0, v^*>0$). For the last two FP all exponents are known.

It turns out that the [3/1] Padé-Borel approximation provides the best estimates for the critical exponents of Ising and polymer models in the 3D case. It is natural to suggest that the same is true for exponents of the dilute model. The ε algorithm of Wynn requires $t_0=0.48$ as a condition for the best fit to the known values. Table 3 contains estimates of the 3D REIM exponents along with some averaged experimental data (Thurston *et al* 1988). Critical exponents derived from conformal Borel analysis of single-variable series (Le Guillou and Zinn-Justin 1980) are included in table 3 for comparison. The Borel summation of the ε expansions to the order $O(\varepsilon^5)$ gives close, though slightly differing, results (Le Guillou and Zinn-Justin 1985).

Other possible approximations to the Borel transformed series, i.e. [1/1], [2/1] and [2/2] (if not excluded due to poles on the positive semi-axis), give close exponent values which could be used to check convergence properties of the Padé-Borel scheme and to define error bounds.

Similar exponent values for the 3D REIM have been obtained recently through Chisholm-Borel analysis of the four-loop expansions: $\gamma=1.326$, $\nu=0.67$, $\alpha=-0.011$ and $\beta=0.342$ (Mayer *et al* 1988). The differences between values obtained by different methods reflect uncertainties in the results.

Table 3. 3D REIM exponents from four-loop expansions. R, I and U stand for random, Ising and unphysical (polymer), respectively. AW indicates the ε algorithm of Wynn ($t_0 = 0.48$) and CB means the conformal mapping Borel technique (Le Guillou and Zinn-Justin 1980).

FP		u^*	v^*	γ	ν	α	β
R	[3/1]	−0.6839	2.2361	1.321	0.6714	−0.013	0.348
	AW	−0.5874	2.1782	1.318	0.6680	−0.004	0.343
	Experiment (Thurston *et al* 1988)			1.37 ± 4	0.70 ± 2	−0.09 ± 7	0.34 ± 1
I	[3/1]	0	1.4299	1.240	0.6301	0.110	0.325
	AW	0	1.4586	1.240	0.6282	0.116	0.322
	CB	0	1.416 ± 5	1.241 ± 2	0.6300 ± 15	0.1100 ± 45	0.3250 ± 15
U	[3/1]	1.4395	0	1.162	0.5895	0.232	0.303
	AW	1.4547	0	1.163	0.5880	0.236	0.301
	CB	1.421 ± 8	0	1.1615 ± 20	0.5880 ± 15	0.2360 ± 45	0.3020 ± 15

The two summation methods are additionally tested by application to exactly solvable 2D models. The critical exponents of the 2D Ising model and 2D polymer limit could be extracted from scaling dimensions of certain operators in conformal invariant theories (Nienhuis 1984, Dotsenko and Fateev 1984) and from Nienhuis's conjectures (Nienhuis 1982). The conjectures give exact critical exponents for a continuous range of the number of components n: $|n| \leq 2$. Putting n equal to 1 or 0 results in Ising or polymer exponents, respectively. The exact and approximated values are compared in table 4 where exponents for the random FP are also included as well as recent experimental data (Hagen *et al* 1987). The 2D REIM is also exactly solvable. Moreover, two different sets of exact results have been published. On the one hand, Dotsenko and Dotsenko (1983) predict $\eta = \beta = 0$, $\gamma = 2$. Their results have been questioned by Shalayev (1984) who has pointed out that the 2D REIM is governed by the same set of exponents as the pure Ising model. The only effect of quenched impurities is to induce logarithmic corrections to the thermodynamic functions (see also Shankar 1987). The present results are consistent with this latter conclusion and with experimental data (for a review of various experiments see Lyuksyutov *et al* (1988)). No indication of a new set of exponents has been found.

Table 4. 2D REIM exponents for four-loop expansions. The notation is the same as for table 3. For AW $t_0 = 0.13$.

FP		u^*	v^*	ν	η	γ
R	[3/1]	−0.0898	1.9866	1.002	0.251	1.753
	AW	−0.1870	2.3835	0.979	0.139	1.821
	Experiment (Hagen *et al* 1987)			1.08 ± 6		1.75 ± 7
I	[3/1]	0	1.8836	0.982	0.250	1.719
	AW	0	2.1666	0.941	0.139	1.751
	Exact			1	$\frac{1}{4}$	$\frac{7}{4}$
U	[3/1]	1.9302	0	0.796	0.235	1.405
	AW	2.0963	0	0.7513	0.110	1.420
	Exact			$\frac{3}{4}$	$\frac{5}{24}$	$\frac{43}{32}$

It follows from the exact solution of the 2D REIM (Shalayev 1984) that there should be no random FP that nevertheless appears in table 4. It could be considered as 'split' from the Ising FP due to the relatively low order of approximation, all the more so since u^* is small as compared with v^*, and suffers substantial variations as the order of approximation is changed (e.g. for the [2/2] Padé-Borel approximation u^* is nearly twice the value for [3/1], while v^* remains about the same).

A further test is provided by the 2D O(n) model with the number of components n formally put equal to -1. The conjectures (Nienhuis 1982) give exact exponents which, along with results obtained in [3/1] and [2/2] Padé-Borel approximations, are included in table 5.

The agreement between approximated and exact values is remarkable.

Table 5. Critical exponents for 2D O(n) ($n = -1$) model.

	ν	η	γ
[3/1]	0.6302	0.1647	1.566
[2/2]	0.6280	0.1432	1.1660
Exact	$\frac{5}{8}$	$\frac{3}{20}$	$\frac{37}{32}$

5. Summary

Four-loop expansions of the coefficients of the Callan-Symanzik equation for the 3D and 2D REIM are presented. To find fixed-point coordinates and to estimate critical exponents two methods of summation of double series are used: a Padé-Borel approximation and the first confluent form of the ε algorithm of Wynn. When projected on one of the variables each method is reduced to its single-variable version.

The order $[L/M]$ of the Padé-Borel approximation is chosen so as to provide the best fit to the single-variable exponents. Exact critical exponents for 2D single-variable problems are known from conformal invariant theories. For 3D models the exponents have been calculated to great accuracy by Le Guillou and Zinn-Justin (1980, 1985). For the ε algorithm of Wynn the free parameter t_0 is fixed. Critical exponents of 2D and 3D dilute models are then estimated within this framework. These procedures give critical exponents of the 2D dilute Ising model coinciding with those of the pure Ising model, in agreement with exact solutions based on the fermionic representation (Shalayev 1984, Shankar 1987). It is seen from table 4 that summation of 2D double expansions gives exponents consistent with exact results. In addition, error bounds for the 2D values are usually much larger than in the 3D case (Le Guillou and Zinn-Justin 1980, 1985). All this suggests that the present estimates for the 3D dilute Ising model should be rather close to the true values.

Experimental data for the systems described by the 3D REIM have relatively large error bars which overlap the range of the present estimates except for the susceptibility exponent γ.

Thus the double expansions for systems with quenched impurities suspected to be Borel non-summable (Bray *et al* 1987) are confirmed to produce results consistent with the exact solution for the 2D REIM and with experimental data for the 3D REIM.

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References

Baker G A Jr and Graves-Morris P 1981 *Padé Approximants* (Reading, MA: Addison-Wesley)
Baker G A Jr, Nickel B G and Meiron D I 1978 *Phys. Rev.* B **17** 1365-74
Bray A J, McCarthy T, Moore M A, Roger J D and Young A 1987 *Phys. Rev.* B **36** 2212-9
Brezinski C 1977 *Accélération de la Convergence en Analyse Numérique* (*Lecture Notes in Mathematics*) (Berlin: Springer)
Brézin E, le Guillou J C and Zinn-Justin J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic) p 125-247
Chisholm J S R 1973 *Math. Comput.* **27** 841-8
Dotsenko Vik S and Dotsenko Vl S 1983 *Adv. Phys.* **32** 129-72
Dotsenko Vl S and Fateev V A 1984 *Nucl. Phys.* B **240** [FS12] 312-48
Grinstein G and Luther A 1976 *Phys. Rev.* B **13** 1329-43
Hagen M, Cowley R A, Nicklow R M and Ikeda H 1987 *Phys. Rev.* B **36** 401-8
Harris A B and Lubensky T C 1974 *Phys. Rev. Lett.* **33** 1540-3
Jug G 1983 *Phys. Rev.* B **27** 609-12
Khmelnitsky D E 1975 *Zh. Eksp. Teor. Fiz.* **68** 1960-8 1976 (*Sov. Phys.-JETP* **41** 981-9)
Le Guillou J C and Zinn-Justin J 1980 *Phys. Rev.* B **21** 3976-98
—— 1985 *J. Physique Lett.* **46** L137-41
Lovitch L and Marziani M F 1983 *Nuovo Cimento A* **76** 615-26
Lubensky T C 1975 *Phys. Rev.* B **11** 3573-80
Lyuksyutov I F, Naumovets A G and Pokrovsky V L 1988 *Two-dimensional Crystals* (Kiev: Naukova Dumka) (in Russian)
Marziani M F 1983 *Lett. Nuovo Cimento* **37** 124-8
—— 1984 *J. Phys. A: Math. Gen.* **17** 547-57
Mayer I O 1988 *Teor. Mat. Fiz.* **75** 234-44
Mayer I O and Sokolov A I 1984 *Fiz. Tverd. Tela* **26** 3454-6
Mayer I O, Sokolov A I and Shalayev B N 1988 *Proc. JSSF-4, Tsukuba, Ferroelectrics* to be published
Newman K E and Riedel E K 1982 *Phys. Rev.* B **25** 264-80
Nickel B G, Meiron D I and Baker G A, Jr 1977 Compilation of 2-pt and 4-pt graphs for continuous spin models *Report* University of Guelph
Nienhuis B 1982 *Phys. Rev. Lett.* **49** 1062-5
—— 1984 *J. Stat. Phys.* **34** 731-61
Sansone G and Gerretsen J 1960 *Lectures on the Theory of Functions of a Complex Variable* vol 1 (Groningen: Noordhoff)
Shalayev B N 1984 *Fiz. Tverd. Tela* **26** 3002-5
Shankar R 1987 *Phys. Rev. Lett.* **58** 2466-9
Thurston T R, Peter C J, Birgeneau R J and Horn P M 1988 *Phys. Rev.* B **37** 9559-63